Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
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Pismeni ispit iz predmeta **Linearna algebra**

Bitna napomena: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. Neka je $\mathcal{V} = \mathbb{R}^n$ i neka je $(a_1, a_2, ..., a_n)^\top$ fiksirani vektor iz $\mathcal{V}$. Dokazati da je familija svih elemenata $(x_1, x_2, ..., x_n)^\top$ iz $\mathcal{V}$ sa osobinom $a_1x_1 + ... + a_nx_n = 0$ vektorski podprostor prostora $\mathcal{V}$. Drugim riječima da je

$$\mathcal{M} = \{(x_1, x_2, ..., x_n)^\top \in \mathcal{V} \,|\, a_1x_1 + ... + a_nx_n = 0\}$$

vektorski podprostor od $\mathcal{V}$. Odrediti bazu i dimenziju ovog podprostora.

2. (a) Neka je φ linearna transformacija $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ takva da $\varphi\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 1$, $\varphi\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} = 1$, $\varphi\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = 0$, $\varphi\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = 0$. Odrediti $\varphi\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$.

(b) Baza vektorskog prostora $\mathcal{L} = \left\{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \,|\, x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0\right\}$ je $\mathcal{B} = \left\{\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\}$. Odrediti mu jedan ortogonalni komplement (u odnosu na standardni unutrašnji (skalarni) proizvod $\langle x, y\rangle = x^\top y$).

3. Neka je T linearni operator na prostoru $\mathbb{R}^2$ koji proizvoljan vektor $v \in \mathbb{R}^2$ preslikava osnom simetrijom s osom u pravoj $y = x$ u vektor v' (vidi sliku). (Drugim riječima T je osna simetrija s osom u pravoj $y = x$).

(a) Odrediti matricu koordinata T u odnosu na standardnu bazu.

(b) Odrediti (koordinate) osnu simetriju tačke $v = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ s osom u pravoj $y = x$.

(c) Odrediti koordinate osne simetrije T (odrediti matricu operatora T) u odnosu na bazu $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$.

4. Zadan je unitarni prostor $\mathrm{Mat}_{2\times2}(\mathbb{R})$ sa skalarnim (unutrašnjim) proizvodom $\langle A, B\rangle = \mathrm{trag}(A^\top B)$ i neka je $\mathcal{L}$ vektorski podprostor $\mathrm{Mat}_{2\times2}(\mathbb{R})$ definiran kao

$$\mathcal{L} = \mathrm{span}\left\{\begin{pmatrix}0 & 3\\3 & -3\end{pmatrix}, \begin{pmatrix}-2 & -2\\6 & -2\end{pmatrix}, \begin{pmatrix}-1 & 1\\-1 & 1\end{pmatrix}\right\}.$$

Nađite ortonormiranu bazu za $\mathcal{L}$.

(#) Neka je $V=\mathbb{R}^n$ i neka je $(a_1, a_2, \dots, a_n)^T$ fiksirani vektor iz V. Dokazati da je familija svih elemenata $(x_1, x_2, \dots, x_n)^T$ iz V sa osobinom $a_1x_1 + \dots + a_nx_n = 0$ vektorski podprostor prostora V. Drugim riječima da je

$$\mathcal{M} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in V \;\middle|\; a_1x_1 + \dots + a_nx_n = 0 \right\}$$

vektorski podprostor od V. Odrediti dimenziju i bazu ovog podprostora.

Rj.

Prisjetimo se:

<u>Neprazan podskup $\mathcal{S}$ vektorskog prostora V je podprostor od V ako i samo ako</u>

<u>(A1) $x, y \in \mathcal{S} \Rightarrow x+y \in \mathcal{S}$;</u>

<u>(M1) $x \in \mathcal{S} \Rightarrow \alpha x \in \mathcal{S}$ za $\forall \alpha \in \mathbb{R}$</u>

Pa pokažimo da vrijede osobine (A1) i (M1).

Izaberimo proizvoljne elemente $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{M}$ i $\alpha \in \mathbb{R}$. Tada

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}.$$

Uvedimo oznake $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M} \iff a_1x_1 + \dots + a_nx_n = 0 \iff a^T x = 0$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{M} \iff a_1y_1 + \dots + a_ny_n = 0 \iff a^T y = 0$$

$$a^T x + a^T y = 0 \iff a^T(x+y) = 0 \Rightarrow x+y = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix} \in \mathcal{M}$$

Prema tome vrijedi (A1)

$$\alpha\cdot x=\alpha\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}\alpha x_1\\ \vdots\\ \alpha x_n\end{pmatrix}$$

$$a^T x=0 \Rightarrow a^T\alpha x=0 \Rightarrow \alpha x\in M \quad \text{vrijedi (M1)}$$

M jest vektorski podprostor

Da bi odredili bazu i dimenziju napišimo M u drugačijem obliku

$$M=\left\{\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in V \;\middle|\; a_1x_1+\dots+a_nx_n=0\right\}=\left\{\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in \mathbb{R}^n \;\middle|\; a^Tx=0\right\}$$

$$=\left\{\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in \mathbb{R}^n \;\middle|\; \begin{bmatrix}a_1 & a_2 & \cdots & a_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}\right\}=$$

$$=\ker\left(\begin{bmatrix}a_1 & a_2 & \cdots & a_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\right).$$

Prema tome $M=\ker(A)$ gdje je $A=\begin{bmatrix}a_1 & a_2 & \cdots & a_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}$.

Znamo da kolone iz opšteg rješenja od $\ker(A)$ formiraju bazu za $\ker(A)$. Pretpostavimo da je $a\neq 0$. Kako je $\operatorname{rang}(A)=1$ ako posmatramo sistem $Ax=0$ to moramo uzeti $n-1$ promjenjivu proizvoljno. Pretpostavimo da je $a_1\neq 0$, kako je $a\neq 0$ to $\exists a_i\neq 0$

$$a_1x_1+a_2x_2+\dots+a_nx_n=0$$

$$x_1=\frac{a_2}{a_1}x_2+\dots+\frac{a_n}{a_1}x_n$$

$$x=\begin{pmatrix}\frac{a_2}{a_1}x_2+\dots+\frac{a_n}{a_1}x_n\\ x_2\\ \vdots\\ x_n\end{pmatrix}$$

Prema tome $\dim(M)=n-1$ i

$$\mathcal{B}=\left\{\begin{pmatrix}\frac{a_2}{a_1}\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix},\begin{pmatrix}\frac{a_3}{a_1}\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix},\dots,\begin{pmatrix}\frac{a_n}{a_1}\\ 0\\ \vdots\\ 0\\ 1\end{pmatrix}\right\}$$

je baza za M.

Neka je φ linearna transformacija $\varphi: \mathbb{R}^4 \to \mathbb{R}$ taka da

$$\varphi\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=1,\quad \varphi\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}=1,\quad \varphi\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}=0,\quad \varphi\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=0.$$

Odrediti $\varphi\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$.

Rj. Prisjetimo se

Linearna transformacija

Neka su $\mathcal{U}$ i $\mathcal{V}$ vektorski prostori nad $\mathbb{R}$. Linearna transformacija T sa $\mathcal{U}$ u $\mathcal{V}$ je linearna f-ja sa $\mathcal{U}$ i $\mathcal{V}$. Drugim riječima

$$T(x+y)=T(x)+T(y) \quad i \quad T(\alpha x)=\alpha T(x) \quad \forall x,y\in\mathcal{U},\ \forall\alpha\in\mathbb{R}$$

Napišimo vektor $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$ kao linearnu kombinaciju vektora $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$, tj. odredi α, β, γ i δ t.d.

$$\alpha\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\beta\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+\delta\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

Nije teško vidjeti da je $-\delta=d \Rightarrow \delta=-d$

$$\beta+\delta=c \Rightarrow \beta=c-\delta \Rightarrow \beta=c+d$$

$$-\beta-\gamma-\delta=b \Rightarrow -c-\cancel{d}-\gamma+\cancel{d}=b \Rightarrow \gamma=-b-c$$

$$\alpha+\beta+\gamma+\delta=a \Rightarrow \alpha=a-\beta-\gamma-\delta=a-\cancel{c}-\cancel{d}+b+\cancel{c}+\cancel{d}=a+b$$

Prema tome $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=(a+b)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+(c+d)\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}+(-b-c)\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+(-d)\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$

Sad nije teško izračunati $\varphi\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

$$\varphi\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a+b)\underbrace{\varphi\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{=1} + (c+d)\underbrace{\varphi\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{=1} + (-b-c)\underbrace{\varphi\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{=0} + (-d)\underbrace{\varphi\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}_{=0}$$

$$= a+b+c+d.$$

(#) Baza vektorskog prostora $\mathcal{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 = 0, -x_1 + 2x_2 + x_3 = 0 \right\}$ je $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\}$. Odrediti mu jedan ortogonalni komplement (u odnosu na standardni unutrašnji (skalarni) proizvod $\langle x, y \rangle = x^T y$).

Rj.

Prisjetimo se

Ortogonalni komplement

Za podskup M unitarnog prostora V, ortogonalni komplement $M^\perp$ od M je definisan sa

$$M^\perp = \{ x \in V \mid \langle m, x \rangle = 0 \text{ za } \forall m \in M \}$$

Primjetimo da je $\mathcal{L} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\} = \operatorname{im} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}_{=A} = \operatorname{im}(A)$.

Teorema ortogonalne dekompozicije

Za svaku matrica $A \in Mat_{m\times n}(\mathbb{R})$

$$\operatorname{im}(A)^\perp = \ker(A^T) \quad i \quad \ker(A)^\perp = \operatorname{im}(A^T)$$

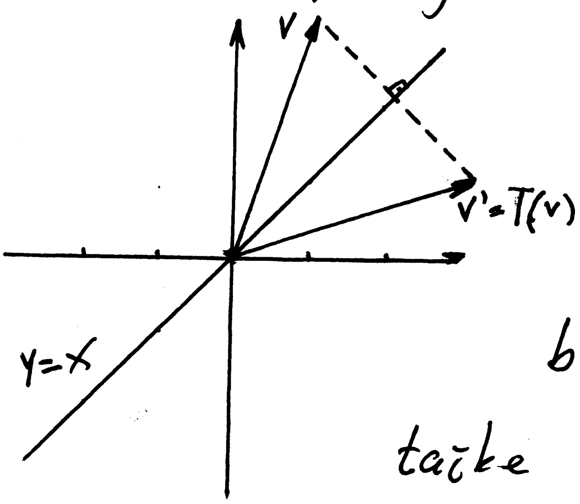
Ortogonalni komplement prostora $\mathcal{L}$ će biti $\ker(\underbrace{[1 \; -1 \; 3]}_{=A^T})$.

$$x - y + 3z = 0$$

$$\begin{matrix} z = s \\ y = t \end{matrix} \Rightarrow x = t - 3s \Rightarrow \begin{pmatrix} t-3s \\ t \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} s$$

Ortogonalni komplement za $\mathcal{L}$ je $\mathcal{L}' = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(#) Neka je T linearni operator na prostoru $\mathbb{R}^2$ koji proizvoljan vektor $v \in \mathbb{R}^2$ preslikava osnom simetrijom s osom u pravoj $y=x$ u vektor v' (vidi sliku).



a) Odrediti matricu koordinata T u odnosu na standardnu bazu.

b) Odrediti (koordinate) osnu simetriju tačke $v=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ s osom u pravoj $y=x$.

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Rj:

a) Prisjetimo se

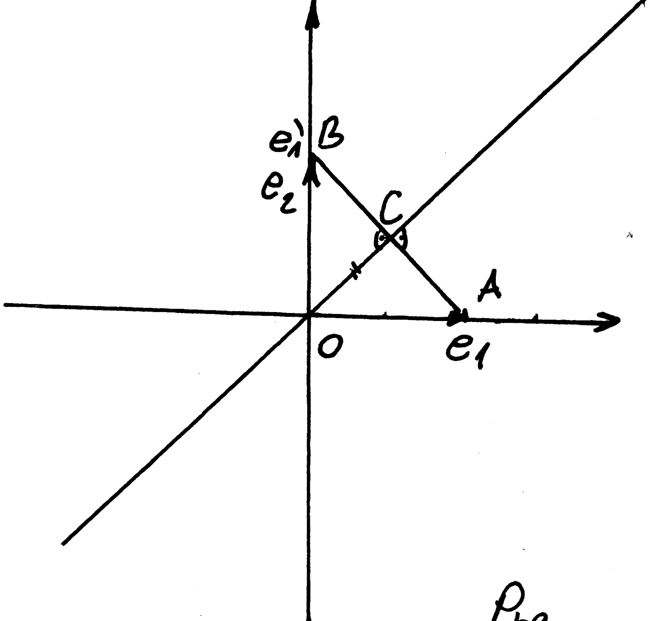
Matrica koordinata

Neka su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_m\}$, redom, baze za $\mathcal{U}$ i $\mathcal{V}$. Matrica koordinata od $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$ u odnosu na par $(\mathcal{B},\mathcal{B}')$ je definisana kao $m\times n$ matrica

$$[T]_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}.$$

Kada je T linearni operator na $\mathcal{U}$, tada je u igri samo jedna baza, i koristimo $[T]_{\mathcal{B}}$ umjesto $[T]_{\mathcal{B}\mathcal{B}}$.

Standardna baza za $\mathbb{R}^2$ je $\mathcal{S}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}=\{e_1,e_2\}$ i da bi odredili $[T]_{\mathcal{S}}$ trebamo pronaći $[T(e_1)]_{\mathcal{S}}$ i $[T(e_2)]_{\mathcal{S}}$.



Pa posmatrajmo kako T djeluje na e_1 i e_2

$T(e_1) = e_1'$.

$$\left.\begin{array}{l} AC \cong CB \\ \angle OCA \cong \angle OCB \\ OC \cong OC \end{array}\right\} \overset{SUS}{\Longrightarrow} \triangle OAC \cong \triangle OCB$$

$$\Downarrow$$

$$\underset{\substack{\| \\ e_1}}{OA} = \underset{\substack{\| \\ e_2}}{OB}$$

Prema tome imamo

$$T(e_1) = e_2 = 0 \cdot e_1 + 1 \cdot e_2$$
$$T(e_2) = e_1 = 1 \cdot e_1 + 0 \cdot e_2$$

$$[T]_\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ tražena matrica koordinata.

b) Prisjetimo se

<u>Djelovanje kao matrično množenje</u>

<u>Neka je $T \in \mathcal{L}(U, V)$, i neka su $\mathcal{B}$ i $\mathcal{B}'$ baze za U i V redom. Za $\forall u \in U$</u>

$$[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} \cdot [u]_{\mathcal{B}}$$

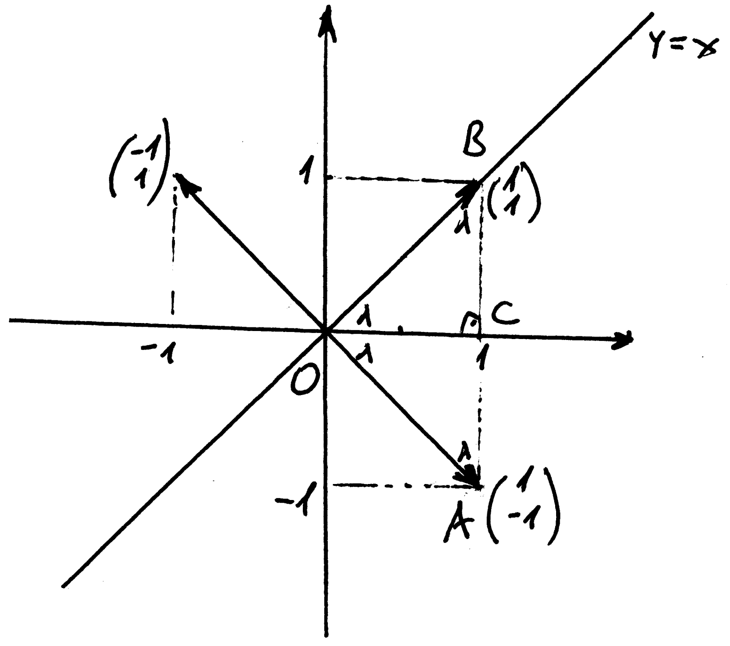
Mi u stvari tražimo $\left[T\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]_\varphi = [T]_\varphi \cdot \left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]_\varphi =$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

Osna simetrija tačke $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ s osom u pravoj $y = x$ je tačka $\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$.

c) Trebamo pronaći $[T]_{\varphi'}$ gdje je $\varphi' = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{e_1'}, \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{e_2'} \right\}$.

$$[T]_{\varphi'} = \begin{pmatrix} | & | \\ \left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\varphi'} & \left[T\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_{\varphi'} \\ | & | \end{pmatrix}.$$



Primjetimo da je vektor $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ okomit na pravu $y=x$. Zašto

OAC jkk pravougli

OCB jkk pravougli

$2\lambda = 90°$

$\lambda = 45°$

Sad primjetimo da je
$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot e_1' + 0 \cdot e_2'$$
$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \cdot e_1' + (-1) \cdot e_2'$$

$$[T]_{\varphi'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ traženo rješenje

Provjera:
$$\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\varphi'} = 1 \cdot e_1' + 0 \cdot e_2' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\varphi'}$$

$$\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_{\varphi'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_{\varphi'}$$

(#) Zadan je unitarni prostor $Mat_{2\times 2}(\mathbb{R})$ sa skalarnim (unutrašnjim) produktom $\langle A, B\rangle = \operatorname{trag}(A^T B)$ i neka je $\mathcal{L}$ vektorski podprostor prostora $Mat_{2\times 2}(\mathbb{R})$ definiran kao

$$\mathcal{L} = \operatorname{span}\left\{\begin{pmatrix} 0 & 3 \\ 3 & -3 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ 6 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right\}$$

Nađite ortonormiranu bazu za $\mathcal{L}$.

Rj. Da bi odredili ortonormiranu bazu za $\mathcal{L}$ koristimo Gram-Schmidtov proces ortonormalizacije.

Klasični Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \displaystyle\sum_{i=1}^{k-1} \langle u_i, x_k\rangle u_i$

$u_k \leftarrow \dfrac{u_k}{\|u_k\|}$

U našem slučaju imamo $X_1 = \begin{pmatrix} 0 & 3 \\ 3 & -3 \end{pmatrix}$, $X_2 = \begin{pmatrix} -2 & -2 \\ 6 & -2 \end{pmatrix}$, $X_3 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$$U_1 \leftarrow \frac{X_1}{\|X_1\|}$$

$$U_2 \leftarrow X_2 - \langle U_1, X_2\rangle U_1$$

$$U_2 \leftarrow \frac{1}{\|U_2\|}\cdot U_2$$

$$U_3 \leftarrow X_3 - \langle U_1, X_3\rangle U_1 - \langle U_2, X_3\rangle U_2$$

$$U_3 \leftarrow \frac{1}{\|U_3\|} U_3$$

$$\|X_1\|^2 = \langle X_1, X_1 \rangle = \operatorname{trag}(X_1^T X_1) = \operatorname{trag}\left(\begin{bmatrix} 0 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 3 & -3 \end{bmatrix}\right) = 9 + 18 = 27$$

$$\|X_1\| = \sqrt{27} = \sqrt{3 \cdot 9} = 3\sqrt{3}$$

$$U_1 \leftarrow \frac{1}{3\sqrt{3}}\begin{pmatrix} 0 & 3 \\ 3 & -3 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\langle U_1, X_2 \rangle = \operatorname{trag}(U_1^T X_2) = \operatorname{trag}\left(\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ 6 & -2 \end{pmatrix}\right) = \frac{1}{\sqrt{3}}(6+0) = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\langle U_1, X_2 \rangle U_1 = \frac{6}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$U_2 \leftarrow \begin{pmatrix} -2 & -2 \\ 6 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 4 & 0 \end{pmatrix} = 2\begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}$$

$$\|U_2\|^2 = \operatorname{trag}\left(4\begin{bmatrix} -1 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 2 & 0 \end{bmatrix}\right) = 4(5+4) = 36 \Rightarrow \|U_2\| = 6$$

$$U_2 \leftarrow \frac{1}{6} \cdot 2\begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}$$

$$\langle U_1, X_3 \rangle = \operatorname{trag}\left(\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right) = \frac{1}{\sqrt{3}}(-1+0) = -\frac{1}{\sqrt{3}}$$

$$\langle U_2, X_3 \rangle = \operatorname{trag}\left(\frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right) = \frac{1}{3}(-1-2) = -1$$

$$\langle U_1, X_3 \rangle U_1 = -\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\langle U_2, X_3 \rangle U_2 = (-1) \cdot \frac{1}{3}\begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$$

$$U_3 \leftarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} = \frac{1}{3}\left[\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix} -4 & 2 \\ 0 & 2 \end{pmatrix}$$

$\Rightarrow \ldots$ Ortonormirana baza za $\mathcal{L}$ je

$$\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}, \frac{\sqrt{6}}{6}\begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$